THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2010F Classwork 4

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Name:

1. (30 points) Study the limit of the following functions at $(0,0)$.

   (a)
$$f(x,y) = \frac{x^2y^2}{|x|+y^2} \ .$$

   (b)
$$g(x,y) = \frac{\sin xy}{x^2+y^2} \ .$$

   (c)
$$h(x,y) = y\log(x^2+|y|) \ .$$

Solution. (a) We have
$$0 \le f(x,y) = \frac{x^2y^2}{|x|+y^2} \le \frac{x^2y^2}{y^2} = x^2 \ .$$
By Sandwich Rule,
$$0 \le \lim_{(x,y)\to(0,0)} f(x,y) \le \lim_{(x,y)\to(0,0)} x^2 = 0 \ ,$$
so
$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \ .$$
(b) Along the sequence $(x_n,0), x_n \downarrow 0, g(x_n,0)| = 0$ hence
$$\lim_{(x_n,0)\to(0,0)} g(x,y) = 0 \ .$$
On the other hand, along $(x_n,x_n), x_n \downarrow 0$,
$$g(x_n,x_n) = \frac{\sin x_n^2}{2x_n^2} \to \frac{1}{2}, \quad n\to\infty \ .$$
So
$$\lim_{(x_n,x_n)\to(0,0)} g(x,y) = \frac{1}{2} \ .$$
There are two sequences converging to $(0,0)$ with different limits, so $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist. (We have used the fact that $\sin t/t \to 1$ as $t\to 0$.)
(c). Indeed, we have
$$\log(|y|) \le \log(|y|+x^2) < 0,$$
when x,y is close enough to $(0,0)$. Then,
$$0 \le |h(x,y)| \le |y||\log(|y|+x^2)| \le |y||\log(|y|)| = |y\log(|y|)|.$$
By Sandwich Rule,
$$0 \le \lim_{(x,y)\to(0,0)} |h(x,y)| \le \lim_{(x,y)\to(0,0)} |y\log(|y|)| = 0,$$

so

$$\lim_{(x,y)\to(0,0)} h(x,y) = 0.$$

We have used the fact that $t\log t \to 0$ as $t \downarrow 0$.

2. (30 points) Find the iterated limits and limit of the function

$$h(x,y) = \frac{x-y}{x+y}$$

at $(0,0)$.

Solution. We first claim that

$$\lim_{y\to 0}\lim_{x\to 0} \frac{x-y}{x+y}$$

dost not exists. Indeed, when $y \neq 0$,

$$\lim_{x\to 0} \frac{x-y}{x+y} = -1.$$

When $y = 0$,

$$\lim_{x\to 0} \frac{x-y}{x+y} = 1$$

So the limit dose not exist.

We can prove that

$$\lim_{x\to 0}\lim_{y\to 0} \frac{x-y}{x+y}$$

dost not exists by the same argument. Consequently, the limit of the function dose not exist.

3. (40 points) Consider the function

$$F(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2} \ .$$

Show that

$$\lim_{y\to 0}\lim_{x\to 0} F(x,y) = \lim_{x\to 0}\lim_{y\to 0} F(x,y) = 0 \ ,$$

but

$$\lim_{(x,y)\to(0,0)} F(x,y)$$

does not exist.

Solution.

$$\lim_{y\to 0}\lim_{x\to 0} F(x,y) = \lim_{y\to 0}(0) = 0$$

$$\lim_{x\to 0}\lim_{y\to 0} F(x,y) = \lim_{x\to 0}(0) = 0$$

Therefore,

$$\lim_{y\to 0}\lim_{x\to 0} F(x,y) = \lim_{x\to 0}\lim_{y\to 0} F(x,y) = 0 \ ,$$

We show $\lim_{(x,y)\to(0,0)} F(x,y)$ does not exist by constructing two sequences converging to $(0,0)$ but with different limit of F: Consider $(x_n, y_n) = (\frac{1}{n}, \frac{1}{n})$. Then $(x_n, y_n) \to (0,0)$ as $n \to \infty$, and $F(x_n, y_n) = \frac{\frac{1}{n^4}}{\frac{1}{n^4}} = 1$, and hence

$$\lim_{n\to\infty} F(x_n, y_n) = 1$$

Consider $(x'_n, y'_n) = (\frac{1}{n}, \frac{2}{n})$. Then $(x_n, y_n) \to (0,0)$ as $n \to \infty$, and $F(x_n, y_n) = \frac{\frac{4}{n^4}}{\frac{5}{n^4}} = \frac{4}{5}$, and hence

$$\lim_{n\to\infty} F(x'_n, y'_n) = \frac{4}{5}$$

Therefore, $\lim_{(x,y)\to(0,0)} F(x,y)$ does not exist.